Abstract Machines, Optimal Reduction, and Streams

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Implementation of lambda-calculus reduction

- **Optimal reduction** was introduced in J.J. Levy’s PhD Thesis in 1978 and defined (by means of sharing graphs) by J. Lamping in 1990.
- **Geometry of Interaction** was introduced by J.-Y. Girard in 1985.
- **Virtual Reduction** (Danos-Regnier 1993) is a way to make GoI close to optimal implementation.
- **Directed Virtual Reduction** (Danos-Pedicini-Regnier 1997) is a modification of the VR to ease implementation.
- **PELCR** (Pedicini-Quaglia 2007) is a complete implementation which permits parallel execution on multi-core machines.
Is a Graph Reduction Technique?

- The **machine state** is represented by a pair:
  - a **dynamic graph**
  - a list of **pending actions**.
- Any **machine transition** is obtained by applying the action $\alpha$ to the current graph $G$, from which we get a pair

$$\alpha.G \rightarrow (\Delta_\alpha, G \cup \{\alpha\}).$$

$\Delta_\alpha = \{\alpha_1, \beta'_1, \ldots, \alpha_m, \beta'_m\}$ is a set of residual actions to be added to pending actions already in the state and $G \cup \{\alpha\}$ is the updated dynamic graph.
More precisely:

Let $A$ be the set of pending actions, so that the couple $(A, G)$ denotes the state of the machine. The transition associated to the action $\alpha \in A$ is then

$$(A, G) \xrightarrow{\tau_\alpha} (A \setminus \{\alpha\} \cup \Delta_\alpha, G \cup \{\alpha\})$$

that is, residual actions $\Delta_\alpha$ are added to the list of pending actions.

The basic computational step is borrowed from half-combustion strategy of DVR it includes a symbolic computation in the algebraic structure associated to the graph (the dynamic monoid) and it is a generalisation of the algebraic computations at the base of Geometry of Interaction.
A computation

The memory of the machine is initialized with an empty graph, so that the execution of a terminating program on the abstract machine is represented by the finite sequence of transitions

$$(A^0, \emptyset) \xrightarrow{\tau_{\alpha_0}} (A^1, G^1) \xrightarrow{\tau_{\alpha_1}} \cdots \xrightarrow{\tau_{\alpha_{n-1}}} (A^n, G^n) \xrightarrow{\tau_{\alpha_n}} (\emptyset, G^{n+1})$$

where $\alpha_i \in A^i$ for all $i = 0, \ldots, n$. Note that the initial action set $A^0$ is the interpretation of the program, and the final graph $G^{n+1}$ represents the result of the evaluation.

In general, the execution of the machine may not terminate and, consequently, be represented by a possibly infinite sequence of elementary steps.
Non termination

When executing on parallel machines, the sequence of residuals produced by non terminating execution on one machine and directed to another one can be infinite.

To cope with this situation we adopt streams as data structures for the pending actions.

Description of the Sequential Abstract Machine, whose setting is reminiscent of SECD machines, employed to give the operational semantics of $\lambda$-calculus.
A bridging model

We introduce a formal description for **multicore “functional” computation** as a step to **quantitatively study** the behaviour of the PELCR implementation.
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As a starting point we assume (what we already know) that PELCR is sound as a “parallel” operational semantics, this means that we do not care on reordering of actions since the computation of the normal form by using Geometry of interaction rules (shared optimal reduction) is local and asynchronous.

Definition (PELCR Actions)

Given a dynamic graph $G = (V, E \subseteq V \times V)$ with edges labeled on the Girard dynamic algebra $\Lambda^*$, we define an action $\alpha$ on $G$ as $\langle \epsilon, e, w \rangle$ where $\epsilon \in \{+,-\}$, $e = (v_t, v_s) \in V^2$ is an edge in $G$ and $w \in \Lambda^*$. 
A bridging model

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We describe the **pelcr virtual machine** (PVM) as an abstract machine working on its state \((C, D)\).

- **\(C\)** contains the **computational task**: a stream of closures (FIFO).
  - A closure is a **signed edge**.
  - A signed edge \(e\) is represented by a triplet \(\langle (\varepsilon_t, \varepsilon_s), (v_t, v_s), w \rangle\), where: \(\varepsilon_t, \varepsilon_s \in \{+, -\}\) are the target and source polarities of \(e\); \(v_t, v_s \in V\) are the target and source nodes of \(e\) and can be considered as their memory addresses; and \(w \in M\) is the label of \(e\).

- **\(D\)** represents the **current memory**, and contains **environment elements**.
  - any environment element has a memory address \(e_i\) and is called **node**.
  - memory at address \(e_i\) contains signed edges.
Girard dynamic algebra is traditionally used in the execution formula of Geometry of Interaction, which is a power series of matrices. Consequently $\Lambda^*$ is considered together with a formal sum of its elements, and consequently it is a (monoid) algebra.
The so-called *Girard dynamic algebra* \( \Lambda^* \) is the dynamic monoid generated by the constants \( p, q \), and a family \( W = \{ w_i \}_i \) of exponential generators, with a morphism \( !(.), \) such that for any \( u \in \Lambda^* \):

\[
\begin{align*}
(\text{annihilation}) \quad x^* y &= \delta_{xy} \quad \text{for } x, y = p, q, w_i, \\
(\text{swapping}) \quad !(u) w_i &= w_i ^{e_i}(u),
\end{align*}
\]

where \( \delta_{xy} \) is the Kronecker operator, \( e_i \) is an integer associated with \( w_i \) called the *lift* of \( w_i \), \( i \) is called the *name* of \( w_i \) and we often write \( w_i, e_i \) to explicitly note the lift of the generator. Notice that swapping and annihilation rules imply that for every \( a, b \in \Lambda^* \) either \( b^* a = 0 \) or it has a stable form, that is \( \Lambda^* \) satisfies SF condition.
For instance, setting \( a = w_{1,2} \) and \( b = !^2 q \), by applying the annihilation rule we get:

\[
b^* a = (!^2 q)^* w_{1,2} = (!q^*) w_{1,2} = w_{1,2}!^2 (!q^*) = w_{1,2} (!^3 q)^* = a' b'^*
\]

with \( a' = a \) and \( b' = !b \).
Example: $(\Delta)$/
The matrix with entries in the Girard dynamic algebra:

\[
\begin{pmatrix}
\text{[ax]}_1 & \text{[ax]}_2 & \text{[ax]}_3 & \text{[ax]}_4 & \text{[cut]}_1 & \text{[t]}_1 \\
0 & 0 & 0 & 0 & q_2 + q_1 & 0 \\
0 & 0 & 0 & 0 & q_1 p + p & 0 \\
0 & 0 & 0 & 0 & q! q + q! p & 0 \\
0 & 0 & 0 & 0 & p & 1 \\
x_2^* q^* + q^* x_1^* q^* & p^* x_1^* q^* + p^* & (q^*) q^* + (p^*) q^* & p^* & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 
\end{pmatrix}
\]
The corresponding graph

Nodes are axioms, cuts and conclusions (terminal nodes):

\[ V = \{ [ax]_1, [ax]_2, [ax]_3, [ax]_4, [cut]_1, [t]_1 \} \]

and edges \((v_t, v_s, w)\) get a weight \(w \in \Lambda^*\) where \(v_t\) is the target node and \(v_s\) is the source node. In this example, the “sparse” representation, consisting of the list of edges with a non-null weight, is more compact:

\[ E = \{ (((cut)_1, [ax]_1), qx_1 q), (((cut)_1, [ax]_1), qx_2), (((cut)_1, [ax]_2), qx_1 p), (((cut)_1, [ax]_2), p), (((cut)_1, [ax]_3), q!q), (((cut)_1, [ax]_3), q!p), (((cut)_1, [ax]_4), p), (((t)_1, [ax]_4), 1) \}. \]
How to control reduction

In a way similar to that of classical SECD machines we define the state of the machine in terms of four components:

- a **stack** $S$, which is used to store the current action;
- an **environment** $E$, is a node of the graph and it provides the local environment where the current action has to be performed;
- a **control** $C$ is the stream of all actions either provided as initial input or created during the execution of other actions, it has to be executed in the context of the graph stored in the memory of the machine;
- a **dump** $D$ corresponds to the current graph and represents the global environment for all future actions.

Transitions are therefore given as

$$(S, E, C, D) \xrightarrow{\tau} (S', E', C', D')$$
0. If \((S, E, C, D) = (\langle \rangle, \text{NULL}, \text{nil}, \emptyset)\) then the machine is in its initial state.
   The initialisation step is
   \[
   (\langle \rangle, \text{NULL}, \text{nil}, \emptyset) \xrightarrow{T} \]

Full Combustion

0. If $(S, E, C, D) = (\langle \rangle, \text{NULL}, \text{nil}, \emptyset)$ then the machine is in its initial state.
   The initialisation step is
   \[(\langle \rangle, \text{NULL}, \text{nil}, \emptyset) \mapsto (\langle \rangle, \text{NULL}, \text{read}, \emptyset).\]
   \text{read}() returns a stream of actions corresponding to the coding of the input in the form of a polarised dynamic graph.
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\text{read()} returns a stream of actions corresponding to the coding of the input in the form of a polarised dynamic graph.

1. If \(v\) is a node and \text{its view is not empty} \(C_v \neq 0\),
0. If \((S, E, C, D) = (\langle \rangle, \text{NULL}, \text{nil}, \emptyset)\) then the machine is in its initial state.

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\[
\begin{align*}
\langle \langle \rangle, \text{NULL}, \text{nil}, \emptyset \rangle \xrightarrow{T} \langle \langle \rangle, \text{NULL}, \text{read()}, \emptyset \rangle. 
\end{align*}
\]

\text{read()} returns a stream of actions corresponding to the coding of the input in the form of a polarised dynamic graph.

1. If \(v\) is a node and \textbf{its view is not empty} \(C_v \neq 0\), we have a reordering of actions \(\sigma\) such that \(C = \sigma(C' \bowtie C_v)\) and \(C' \approx \sigma(C' \bowtie 0)\), then
0. If \((S, E, C, D) = (\langle\rangle, \text{NULL}, \text{nil}, \emptyset)\) then the machine is in its initial state.

The initialisation step is

\[
(\langle\rangle, \text{NULL}, \text{nil}, \emptyset) \xrightarrow{\tau} (\langle\rangle, \text{NULL, read}(), \emptyset).
\]

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1. If \(v\) is a node and \textbf{its view is not empty} \(C_v \neq 0\), we have a reordering of actions \(\sigma\) such that \(C = \sigma(C' \times C_v)\) and \(C' \approx \sigma(C' \times 0)\), then

\[
(\langle\rangle, \text{NULL, } C, D) \xrightarrow{\tau}
\]
0. If \((S, E, C, D) = (\langle \rangle, \text{NULL}, \text{nil}, \emptyset)\) then the machine is in its initial state. The initialisation step is

\[(\langle \rangle, \text{NULL}, \text{nil}, \emptyset) \xrightarrow{\tau} (\langle \rangle, \text{NULL}, \text{read()}, \emptyset).\]

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\[(\langle \rangle, \text{NULL}, C, D) \xrightarrow{\tau} (C_v, \{v\}, \emptyset, C', D)\]
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(\langle \rangle, \text{NULL}, C, D) \xrightarrow{T} (C_v, (\{v\}, \emptyset), C', D)
\]

2. If \(S = \alpha :: S', E = (\{v\}, Y)\), and \(\alpha = \langle(e, e_s), (v, v_s), w\rangle\)
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   \[
   (\langle \rangle, \text{NULL}, C, D) \xrightarrow{\tau} (C_v, (\{v\}, \emptyset), C', D)
   \]

2. If \(S = \alpha :: S', E = (\{v\}, Y)\), and \(\alpha = \langle (\varepsilon, \varepsilon_s), (v, v_s), w \rangle\) then
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   (S, E, C, D) \xrightarrow{\tau}
   \]
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1. If $v$ is a node and its view is not empty $C_v \neq 0$, we have a reordering of actions $\sigma$ such that $C = \sigma(C' \ltimes C_v)$ and $C' \approx \sigma(C' \ltimes 0)$, then
   \[ (\langle \rangle, \text{NULL}, C, D) \xrightarrow{\tau} (C_v, (\{v\}, \emptyset), C', D) \]

2. If $S = \alpha :: S'$, $E = (\{v\}, Y)$, and $\alpha = \langle (\varepsilon, \varepsilon_s), (v, v_s), w \rangle$ then
   \[ (S, E, C, D) \xrightarrow{\tau} (S', (\{v\}, Y \cup \{\alpha\}), C \ltimes \text{execute}_v(\alpha), D), \]

3. When the stack is empty we set the environment to NULL:
   \[ (\langle \rangle, E, C, D) \xrightarrow{\tau} (\langle \rangle, \text{NULL}, C, D) \]
   and continue with step 1.
Synchronous Machine

0 read from input stream

\[(0 \otimes 0, \text{NULL} \otimes \text{NULL}, \text{nil} \otimes \text{nil}, \emptyset \otimes \emptyset) \mapsto (0 \otimes 0, \text{NULL} \otimes \text{NULL}, \text{read()} \otimes \text{nil}, \emptyset \otimes \emptyset)\]

1 actions \(\alpha_1\) and \(\alpha_2\) are synchronously extracted from streams \(C_1\) and \(C_2\)

\[(0 \otimes 0, \text{NULL} \otimes \text{NULL}, \alpha_1 :: C'_1 \otimes \alpha_2 :: C'_2, D_1 \otimes D_2) \mapsto (\alpha_1 \otimes \alpha_2, \text{NULL} \otimes \text{NULL}, C'_1 \otimes C'_2, D_1 \otimes D_2)\]
Simultaneous environment access for both actions:

\[(\alpha_1 \otimes \alpha_2, \text{NULL} \otimes \text{NULL}, C_1 \otimes C_2, D_1 \otimes D_2) \mapsto (\alpha_1 \otimes \alpha_2, v_t^1 \otimes v_t^2, C_1 \otimes C_2, D'_1 \otimes D'_2)\]

when \(\alpha_i = \langle \epsilon_i, e_i, w_i \rangle\) and either \(e_i = (v_t^i, v_s^i)\) or \(v_t^i\) is undefined if \(\alpha_i = 0\) then

\[D'_i = \begin{cases} D_i & \text{if } v_t^i \text{ already is a node of } D_i, \\ D_i \cup \{v_t^i\} & \text{if } v_t^i \text{ is a new node to be added to } D_i. \end{cases}\]

Actions execution

\[(\alpha_1 \otimes \alpha_2, v_t^1 \otimes v_t^2, C_1 \otimes C_2, D_1 \otimes D_2) \mapsto (0 \otimes 0, \text{NULL} \otimes \text{NULL}, (C_1 \otimes \text{execute}_1(\alpha_1)) \otimes \text{execute}_1(\alpha_2)) \otimes (C_2 \otimes \text{execute}_2(\alpha_1)) \otimes \text{execute}_2(\alpha_2), D'_1 \otimes D'_2)\]

The graph \(D'_i = D_i \cup ((v_t^i, v_s^i)^{\epsilon_i}), w_i)\).
Aynchronous Machine

The state of the asynchronous machine is annotated with the scheduled processing unit:

\[(p, S, E, C, D) = (p, S_1 \otimes S_2, E_1 \otimes E_2, C_1 \otimes C_2, D_1 \otimes D_2)\]

where \(p \in \{1, 2\}\) is the order number of the scheduled processor.

The sequence of controls \(p\) is by itself a stream (of integers \(\{1, 2\}\)). We may either choose a random sequence or we may force a particular scheduling by explicitly giving it.
Asynchronous parallel SECD

0 **reading** from the input interface:

\((1, 0 \otimes 0, \text{NULL} \otimes \text{NULL}, \text{nil} \otimes \text{nil}, \emptyset \otimes \emptyset) \mapsto (1, 0 \otimes 0, \text{NULL} \otimes \text{NULL}, \text{read()} \otimes \text{nil}, \emptyset \otimes \emptyset)\)

1 action \(\alpha_p\) **extraction from the stream** \(C_p\):

\((p, S_1 \otimes S_2, E_1 \otimes E_2, C_1 \otimes C_2, D_1 \otimes D_2) \mapsto (p', S_1' \otimes S_2', E_1' \otimes E_2', C_1' \otimes C_2', D_1' \otimes D_2')\)

if \(S_p = 0, E_p = \text{NULL}, C_p = \alpha_p :: C_p'\) then

\[S_i' = \begin{cases} S_i & \text{if } i \neq p \\ \alpha_i & \text{if } i = p \end{cases} \]

\(E_i' = E_i, C_i' = C_i \) if \(i \neq p\) and \(D_i' = D_i\), finally \(p'\) is taken in accord to the scheduling function.
Asynchronous parallel SECD (cont.)

2 action $\alpha_p$'s **environment access**:

$$(p, S_1 \otimes S_2, E_1 \otimes E_2, C_1 \otimes C_2, D_1 \otimes D_2) \mapsto (p', S_1' \otimes S_2', E_1' \otimes E_2', C_1' \otimes C_2', D_1' \otimes D_2')$$

when $S_p = \alpha_p = \langle \epsilon_p, e_p, w_p \rangle$, where

$$E_i' = \begin{cases} E_i & \text{if } i \neq p \\ v^p_t & \text{if } i = p \end{cases}$$

$S_i' = S_i, C_i' = C_i$ and

$$D_i' = \begin{cases} D_i & \text{if } i \neq p \text{ or } i = p \text{ and } v^p_t \in D_p, \\ D_i \cup \{v^p_t\} & \text{if } i = p \text{ and } v^p_t \notin D_p. \end{cases}$$
Asynchronous parallel SECD (cont.)

**3 action execution:**

\[(p, S_1 \otimes S_2, E_1 \otimes E_2, C_1 \otimes C_2, D_1 \otimes D_2) \mapsto (p', S'_1 \otimes S'_2, E'_1 \otimes E'_2, C'_1 \otimes C'_2, D'_1 \otimes D'_2)\]

when \(S_p = \alpha_p = \langle \epsilon_p, e_p, w_p \rangle\), \(E_p = v^p_t\), then

\[S'_i = \begin{cases} S_i & \text{if } i \neq p \\ 0 & \text{if } i = p \end{cases} \quad E'_i = \begin{cases} E_i & \text{if } i \neq p \\ \text{NULL} & \text{if } i = p \end{cases}\]

\[C'_i = C_i \otimes \text{execute}_i(\alpha_p)\]

and the graph \(D'_i = D_i\) for all \(i \neq p\) and \(D'_p\) is obtained from \(D_p\) by adding the edge \(((v^p_t, v^p_s)^\epsilon_p, w_p)\).
The parallel machine can read from two different channels and execute "in parallel" reading the interpretation of two terms:

$$\langle \langle \rangle, \text{NULL}, \text{nil}, \emptyset \rangle \overset{T}{\rightarrow}$$
New ideas to extend the computation

The parallel machine can read from two different channels and execute "in parallel" reading the interpretation of two terms:

\[(\langle \rangle, \text{NULL}, \text{nil}, \emptyset) \xrightarrow{T} (\langle \rangle, \text{NULL}, \text{read}(0) \oplus \text{read}(1), \emptyset).\]

the only missing part is now the way we can read-back the two results which are independently evaluated by the machine.

Execution in the asynchronous version can now be extended to execute non-deterministic within a processor one of the two terms, so schedules are of type \(p.b\) wher \(p\) is the processor and \(b\) is one of the two subterms.